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THE SINGULARITIES OF THE 3-SECANT CURVE ASSOCIATED TO A SPACE CURVE

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ABSTRACT. Let C be a curve in P^3 over an algebraically closed field of characteristic zero. We assume that C is nonsingular and contains no plane component except possibly an irreducible conic.

In [**GP**] one defines closed r-secant varieties to C, $r \in N$. These varieties are embedded in G, the Grassmannian of lines in P^3. Denote by T the 3-secant variety (curve), and assume that the set of 4-secants is finite. Let $\tilde{T}$ be the curve obtained by blowing up the ideal of 4-secants in T. The curve $\tilde{T}$ is in general not in G.

We study the local geometry of $\tilde{T}$ at any point whose fibre of the blowing-up map is reduced at the point. The multiplicity of $\tilde{T}$ at such a point is determined in terms of the local geometry of C at certain chosen secant points. Furthermore we give a geometrical interpretation of the tangential directions of $\tilde{T}$ at a singular point. We also give a criterion for whether all the tangential directions are distinct or not.

1. Introduction. In this paper we consider a nonsingular curve C in P^3_K, where K is an algebraically closed field of characteristic 0. We assume that C contains no plane component except possibly an irreducible conic.

There are various ways of studying the scheme of 3-secants to C. In [**La1**] one defines secant schemes from a functorial point of view (see also [**La2**] where a generalized trisecant lemma is given). In this paper we follow the approach of [**GP**]. There one studies a curve T in the Grassmannian of lines in P^3 whose points correspond to lines intersecting C at least three times, counted with multiplicity.

We will always assume that the set of 4-secants is finite. Let $\tilde{T}$ denote the curve obtained by blowing up T in the scheme of 4-secants. The goal of this paper is to study the local nature of $\tilde{T}$. If a fibre of this blowing-up map is reduced at a point, the dimension of the tangent space of $\tilde{T}$ at this point is at most 2. We give conditions on C that determine the multiplicity of $\tilde{T}$ at such a point (Theorem 2.3.1), and we give a geometrical interpretation of the tangential directions of $\tilde{T}$ at the point (Theorem 2.3.2). We also give a criterion for whether a singularity of $\tilde{T}$ is ordinary or not (Theorem 2.3.3). These results are stated without proofs in §2.

In §3 we consider 3-secants that are not 4-secants, and we sketch a proof of the results of §2 in this case. We show in §4 that at any point of T, contained in a fibre of the blowing-up map which is reduced at this point, we are essentially in a situation covered by the discussion in §3.

In §5 we give an application of Theorems 2.3.1, 2.3.2 and 2.3.3 in the case where C is a smooth complete intersection of two cubic surfaces in P^3.

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In [**WL**] Tore Wentzel-Larsen studies the local geometry of secant varieties using the set-up of [**La1**] where a 3-secant curve $\mathrm{Sec}_3(C)$ is defined, which corresponds bijectively with our curve $\tilde{T}$. The results in [**WL**] partly overlap with those in this paper. In particular [**WL**] contains the same multiplicity formula as the one in our Theorem 2.3.1.

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2. Definitions and main results.

2.1. Let G be the Grassmannian of lines in P^3. Given a line $L \subset P^3$, we denote by $l(L)$ (or l) the corresponding point in G. Let

$$F = \{(P, l(L)) \in P^3 \times G | P \in L\}.$$

Let $p\colon F \to P^3$ and $q\colon F \to G$ be the natural projection maps, and set $\mathcal{C} = p^{-1}(C)$.

The j-secant scheme associated to C is defined by $F^{j-1}(q_*\mathcal{O}_{\mathcal{C}})$, the $(j-1)$th Fitting ideal of the $\mathcal{O}_G$-module $q_*\mathcal{O}_{\mathcal{C}}$. For a variety X over K, and an $\mathcal{O}_X$-module $\mathcal{F}$ we recall that

$$V(F^r(\mathcal{F})) = \{x \in X | \dim_{K(x)}(\mathcal{F} \otimes K(x)) > r\},$$

and that $X \backslash V(F^r(\mathcal{F}))$ is the largest open subscheme, where $\mathcal{F}$ can be generated locally by r elements.

We then see that the support of $F^{j-1}(q_*\mathcal{O}_{\mathcal{C}})$ is $\{l(L) | \mathrm{rk}(\mathcal{O}_C \otimes_{\mathcal{O}_{P^3}} \mathcal{O}_L) \geq j\}$. We denote the 3-secant scheme by T. By the trisecant lemma (see e.g. [**M**, **Ab**, **An**, **Sa**]) T is a curve in G. We always assume that the scheme of 4-secants is finite. Let $\phi\colon \tilde{T} \to T$ denote the blowing-up of the sheaf of ideals $F^3(q_*\mathcal{O}_{\mathcal{C}})\mathcal{O}_T$.

2.2. Assume $\mathrm{rk}(\mathcal{O}_C \otimes_{\mathcal{O}_{P^3}} \mathcal{O}_L) = n \geq 3$ and $L \cap C = \{P_1, \dots, P_k\}$. Let

$$n_i = \mathrm{rk}\left(\mathcal{O}_C \otimes_{\mathcal{O}_{P^3}} \mathcal{O}_L\right)_{P_i}, \qquad i = 1, \dots, k.$$

Then $n = \sum_{i=1}^k n_i$. It is known [**GP**, p. 21] that the points of $\tilde{T}$ in the fibre of ϕ over $l(L)$ are in one-to-one correspondence with the set of k-tuples $(n_{1x}, \dots, n_{kx})$ such that $\sum_{i=1}^k n_{ix} = 3$, and such that $0 \leq n_{ix} \leq n_i$ for $i = 1, \dots, k$. The notation $(n_{1x}, \dots, n_{kx})$ is explained in [**GP**, p. 21]. Briefly each point $\phi^{-1}(l(L))$ corresponds to an $x = (x_0, \dots, x_{n-4}) \in K^{n-3}$ such that $\Omega_x = x_0 + x_1 Z_1 + \cdots + x_{n-4} Z^{n-4} + Z^{n-3}$ divides $\prod_{i=1}^k (Z - z_i)^{n_i}$ in $K[Z]$, where $P_i = (0, 0, z_i, 1)$, $i = 1, \dots, k$. We have

$$\prod_{i=1}^k (Z - z_i)^{n_{ix}} = \prod_{i=1}^k (Z - z_i)^{n_i} / \Omega_x.$$

We denote the point in $\phi^{-1}(l(L))$ corresponding to x, or $(n_{1x}, \dots, n_{kx})$, by l_x. For a curve X and a surface F in P^3 we denote by $I(Q, X \cap F)$ the intersection number of X and F at the point Q. The following is well known.

PROPOSITION 2.2.1 [**GP**, PROPOSITION 3.4]. *$\tilde{T}$ is singular at a point l_x iff there exists a plane H such that $I(P_i, C \cap H) \geq 2n_{ix}$ for $i = 1, \dots, k$.*

We shall show in §4 that for a point l_x in the fibre $\phi^{-1}(l)$, the fibre $\phi^{-1}(l)$ is reduced at l_x iff in the corresponding k-tuple $(n_{1x}, \dots, n_{kx})$ we have that n_{ix} is either 0 or n_i for $i = 1, \ldots, k$. When $\tilde{T}$ is singular at such a point, the plane H described in Proposition 2.2.1 is uniquely determined.

To see this we study the three possible cases:

Case 1: $n_{1x} = n_{2x} = n_{3x} = 1$. H must contain L and the tangent line to C at say P_1. This tangent line is not equal to L because that would imply $n_1 \geq 2$.

Case 2: $n_{1x} = 2$, $n_{2x} = 1$. H must contain L and the tangent line to C at P_2. By the same argument as in Case 1 this tangent line is not equal to L.

Case 3: $n_{1x} = 3$. H must intersect C at least six times at P_1, i.e. H must pass through P and contain the jth derivative vector of C at P_1, $j = 1, \ldots, 5$, for a suitable local parametrization of C at P_1. Since $n_1 = 3$, the first and second derivative vectors are contained in L, while the third derivative vector is not.

We then have that the first and third derivative vectors of C at P_1 span a unique plane.

2.3. Throughout this section we assume that a point $l_x \in T$ is contained in the fibre $\phi^{-1}(l(L))$ of the blowing-up map, that $\phi^{-1}(l(L))$ is reduced at the point l_x, and that $L \cap C = \{P_1, \ldots, P_k\}$. We also assume that $\tilde{T}$ is singular at l_x. Let m_i be the intersection number $I(P_i, C \cap H)$ for $i = 1, \ldots, k$, where H is the special plane described in Proposition 2.2.1. Then we have the following.

THEOREM 2.3.1. *The dimension of the tangent space of $\tilde{T}$ at l_x is 2, and the multiplicity of $\tilde{T}$ at l_x is m, where*

$$m = \min_{i, n_{ix} \neq 0} [m_i/n_i].$$

($[x]$ means the integral part of the real number x.)

DEFINITION. Let X be a variety, P a point of X and $\mathfrak{m}$ the maximal ideal of the local ring $\hat{\mathcal{O}}_{X,P}$. Then the tangent cone of X at P is defined as

$$\mathcal{T}_P(X) = \operatorname{Spec}\left(\bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}\right).$$

The projectivized tangent cone of X at P is defined as

$$P\mathcal{T}_P(X) = \operatorname{Proj}\left(\bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}\right).$$

By Theorem 2.3.1 we can embed $\mathcal{T}_{l_x}(\tilde{T})$ in a plane as m lines (counted with multiplicity) through a point.

We show in §4, (a) and (b), that in our situation there is a natural isomorphism

$$\psi \colon \hat{\mathcal{O}}_{\tilde{T}, l_x} \simeq \mathcal{O}_{G, l(L)}/\mathfrak{A}_x \quad \text{for some ideal } \mathfrak{A}_x.$$

Hence we can embed $\mathcal{T}_{l_x}(\tilde{T})$ as m lines through $l(L)$ in (an affine part of) some plane in P^5 via ψ and the Plücker embedding $G \subset P^5$.

For a point l_x satisfying the assumptions at the beginning of §2.3 the isomorphism ψ corresponds to "forgetting what happens outside those P_i such that $n_{ix} = n_i$, and thus pretend that L is a 3-secant but not a 4-secant". ψ reduces to the obvious isomorphism $\hat{\mathcal{O}}_{\tilde{T}, l_x} \simeq \hat{\mathcal{O}}_{T, l(L)}$ when L is a 3-secant, but not a 4-secant, and the embedded (compactified) tangent cone in P^5 is simply the union of the m-tangents to T at $l(L)$.

All of §4 is devoted to giving a precise formulation of the last, rather loose statements, and to proving the precise statements (i.e. (a) and (b) at the beginning of §4).

Let $\mathcal{L}_1, \dots, \mathcal{L}_m$ be the m (not necessarily distinct) lines in P^5 determined by the embedding of $\mathcal{T}_{l_x}(\tilde{T})$ via ψ. We recall the H is the special plane described in Proposition 2.2.1.

DEFINITION. $\check{H}$ is the plane in $G \subset P^5$, such that the points of H parametrize the lines in $H \subset P^3$.

We will show that each of the lines $\mathcal{L}_1, \dots, \mathcal{L}_m$ are contained in $\check{H}$. By plane duality we then have that to a line $\mathcal{L}$ in $\check{H}$ through the point $l(L)$ there corresponds a unique point $Q \in L \subset P^3$, such that the points of $\mathcal{L}$ parametrize the lines in H through Q.

DEFINITION. We say that a point $Q \in L \subset P^3$ corresponds H-dually to a tangential direction of T at l_x if one of the lines $\mathcal{L}_1, \dots, \mathcal{L}_m$, say $\mathcal{L}_i$, is contained in $\check{H}$, and if the points of $\mathcal{L}_i$ parametrize the lines in H through the point Q.

If $\mathcal{L}_i$ appears at least twice among $\mathcal{L}_1, \dots, \mathcal{L}_m$, we say that Q corresponds H-dually to a multiple tangential direction of $\tilde{T}$ at l_x.

Under the assumptions made at the beginning of §2.3 we now give a closer description of the tangential directions of $\tilde{T}$ at a point l_x. Let the multiplicity of T at l_x be m.

THEOREM 2.3.2. (i) *A tangential direction of $\tilde{T}$ at l_x always corresponds H-dually to some point $Q \in L$.*

(ii) *$Q \in L$ corresponds H-dually to a tangential direction iff there exists a surface M in P^3 such that*

(a) *$\deg M = m + 1$, and M has a singularity of order at least m at Q.*

(b) *$I(P_i, C \cap M) \geq (m+1)n_{ix}$ for $i = 1, \dots, k$.*

(c) *$L \not\subset \operatorname{Sing}(M)$, and $m \cdot L \subset M \cap H$, i.e. $I(M) \subset (I(H) + I(L)^m)$.*

(d) *Modulo the square of the ideal defining L, the equation defining M is equal to the equation of a cone of degree $m+1$ with vertex at Q.*

DEFINITION. If a curve X is singular at a point P, we say that the singularity is nonordinary if the projectivized tangent cone $P\mathcal{T}_p(X)$ is singular (i.e. if it contains a multiple point).

THEOREM 2.3.3. *The singularity of $\tilde{T}$ at the point l_x is nonordinary with $Q \in L$ corresponding H-dually to a multiple tangential direction iff there exists a surface N such that*

(a) *N is a cone of degree m with vertex at Q.*

(b) *$I(P_i, C \cap N) \geq (m+1)n_{ix}$ for $i = 1, \dots, k$.*

(c) *$L \not\subset \operatorname{Sing}(N)$.*

REMARK. Theorem 2.3.3 says that a tangential direction is multiple iff some surface M satisfying the properties of Theorem 2.3.2 breaks up into the union of a cone of degree m and a plane not containing any of the points of $L \cap C$.

3. Proof of the main results for 3-secants that are not 4-secants.

3.1. If $l_x \in \tilde{T}$ is the unique point in $\phi^{-1}(l(L))$, where L is a 3-secant which is not a 4-secant, then $\hat{\mathcal{O}}_{\tilde{T}, l_x}$ is isomorphic to $\hat{\mathcal{O}}_{T,l}$. In this situation we prefer to work with T, hence in G where the computations have natural geometric interpretations.

We shall use the notation of [**GP**, pp. 15–16], and consider first a secant L such that $\operatorname{rk}(\mathcal{O}_C \otimes_{\mathcal{O}_{P^3}} \mathcal{O}_L) = n \geq 3$. (This notation will be used also in §4.) We choose coordinates X', Y', Z', W' for P^3, such that L has equations $X' = Y' = 0$, and such that $L \cap C$ is contained in the affine space A^3 with equation $W' \neq 0$. Let A^3 be $\operatorname{Spec} K[X, Y, Z]$, where $X = X'/W', Y = Y'/W', Z = Z'/W'$. We set $L \cap C = \{P_1, \dots, P_k\}$, where $P_i = (0, 0, z_i)$ for $i = 1, \dots, k$, and choose the plane $Z = 0$ such that

$$X = \sum_{j \geq n_i} \alpha_{i,j}(Z - z_i)^j, \qquad Y = \sum_{j \geq n_i} \beta_{i,j}(Z - z_i)^j$$

are local parametrizations of C at P_i, where n_i is $\operatorname{rk}(\mathcal{O}_C \otimes_{\mathcal{O}_{P^3}} \mathcal{O}_L)_{P_i}$ for $i = 1, \dots, k$. For each $P_i \in L \cap C$ there is a unique plane H_i containing L, such that

$$I(P_i, C \cap H_i) \geq n_i + 1.$$

One uses the same argument as at the end of §2.2 to prove this. We can assume that none of these finitely many planes has equation $X = 0$, i.e. we assume $\alpha_{i,n_i} \neq 0$ for $i = 1, \dots, k$. We assume that $\tilde{T}$ is singular at the point l_x, and that the fibre $\phi^{-1}(l(L))$ is reduced at l_x. We choose $Y = 0$ as the equation of the unique plane H described in Proposition 2.2.1.

Let R be $\hat{\mathcal{O}}_{G,l}$. We recall the point-line incidence variety F introduced in §2.1. There exists a regular system of parameters (a, b, c, d) of R such that the inverse image of F in $A^3 \times \operatorname{Spec} R = \operatorname{Spec} R[X, Y, Z]$ is defined by $X = a + bZ$, $Y = c + dZ$. The completion of the local ring of $\mathcal{C} = p^{-1}(C)$ at (l, P_i) is $A_i = R[[Z - z_i]]/\mathfrak{A}_i$, where $\mathfrak{A}_i$ is generated by

$$f_i = a + bZ - \sum_{j \geq n_i} \alpha_{i,j}(Z - z_i)^j, \qquad g_i = c + dZ - \sum_{j \geq n_i} \beta_{i,j}(Z - z_i)^j$$

for $i = 1, \dots, k$. When $n = 3$, we have

$$F^2(q_* \mathcal{O}_{\mathcal{C}})\hat{\mathcal{O}}_{G,l} = F^2\left(\bigoplus_{i=1}^{k} A_i\right)\hat{\mathcal{O}}_{G,l} = \sum_{i=1}^{k} F^{n_i - 1}(A_i)\hat{\mathcal{O}}_{G,l}.$$

The last equality follows from the general

$$F^{r-1}\left(\bigoplus_{i=1}^{k} A_i\right) = \sum_{j_1 + \dots + j_k = r-1} F^{j_1}(A_1) \times \dots \times F^{j_k}(A_k)$$

(see e.g. [**GP**, p. 16]). When $r = \sum_{i=1}^{k} n_i = 3$, the right-hand side reduces to $\sum_{i=1}^{k} F^{n_i - 1}(A_i)$ since the $\hat{\mathcal{O}}_{G,l}$-module A_i is generated by n_i elements for $i = 1, \dots, k$.

By Weierstrass' Preparation Theorem [**ZS**, pp. 140–141, 145], there is a distinguished polynomial S_i of degree n_i in $Z - z_i$ such that S_i generates the same ideal as f_i in $R[[Z - z_i]]$. There is also a polynomial T_i of degree $n_i - 1$ with coefficients in $\mathcal{M} = (a, b, c, d)$, such that modulo S_i we have $T_i \equiv g_i$. The coefficients of the T_i generate the ideal

$$\sum_{i=1}^{k} F^{n_i - 1}(A_i)\hat{\mathcal{O}}_{G,l}.$$

To see this one uses the resolution

$$R[[Z-z_i]]/S_i \xrightarrow{\cdot T_i} R[[Z-z_i]]/S_i \to R[[Z-z_i]]/(S_i,T_i) \to 0.$$

We obtain a matrix description of the map to the left by choosing $(Z_1-z_i)^j$, $j = 0,\dots,n_i-1$, as an R-basis of $R[[Z-z_i]]/S_i$ on both sides. Then the first column of the matrix consists of the coefficients of T_i, and the other entries of the matrix are contained in the ideal generated by the entries in the first column. Hence when $n=3$ and $\phi(l_x)=l(L)$, we have

$$\hat{\mathcal{O}}_{\tilde{T},l_x} \simeq \hat{\mathcal{O}}_{T,l(L)} \simeq R/(\text{the coefficients of the } T_i).$$

We have 3 different cases when $n=3$:

Case 1. $L\cap C=\{P_1,P_2,P_3\}$, $n_1=n_2=n_3=1$.

Case 2. $L\cap C=\{P_1,P_2\}$, $n_1=2, n_2=1$.

Case 3. $L\cap C=\{P_1\}$, $n_1=3$.

We prove our main results in Case 1. The proofs of the main results in the two other cases follow the same lines and are contained in [**J**].

3.2. PROOF OF THEOREM 2.3.1 IN CASE 1. Let $m=\min_i m_i$, where $m_i=I(P_i,C\cap H)$, $i=1,\dots,k$, and H is the special plane described in Proposition 2.2.1. Since the equation of H is $Y=0$, m_i is $\min\{j|\beta_{ij}\neq 0\}$. We will show that the multiplicity of T at l is m. Using Weierstrass' Preparation Theorem [**ZS**, pp. 140–141, 145] we obtain

$$S_i = Z - z_i + s_{i,0}(a,b), \quad \text{where } s_{i,0} \equiv -\frac{a+z_ib}{\alpha_{i,1}} \text{ modulo}(a,b)^2,$$

$$T_i \equiv c + z_id - \left(\frac{a+z_ib}{\alpha_{i,1}}\right)^m \beta_{i,m} \quad \text{modulo}((a,b)\cdot(c,d),(a,b,c,d)^{m+1}).$$

We have $\hat{\mathcal{O}}_{T,l} = R/(T_1,T_2,T_3)$, where $R=k[[a,b,c,d]]$. We denote by $\bar{a},\bar{b},\bar{c},\bar{d}$ the images of a,b,c,d in $\hat{\mathcal{O}}_{T,l}$. Let $\mathcal{M}=(\bar{a},\bar{b},\bar{c},\bar{d})$. It is clear that $\mathcal{M}=(\bar{a},\bar{b})$. A little calculation gives that the unique relation between $\bar{a}$ and $\bar{b} \bmod \mathcal{M}^{m+1}$ is

$$(*) \qquad \sum_{j=0}^{m}\binom{m}{j}\gamma_j\bar{a}^{m-j}\bar{b}^j \equiv 0,$$

where

$$\gamma_j = \frac{\beta_{1,m}}{\alpha_{1,1}^m}(z_2-z_3)z_1^j + \frac{\beta_{2,m}}{\alpha_{2,1}^m}(z_3-z_1)z_2^j + \frac{\beta_{3,m}}{\alpha_{3,1}^m}(z_1-z_2)z_3^j.$$

This implies that the multiplicity of T at l is at least m since the unique relation between a and $\bar{b}$ modulo $\mathcal{M}^{m+1}$ contains no term of degree less than m.

To prove that the multiplicity is exactly m we must show that the relation $(*)$ does not vanish identically, i.e. we must show that some γ_j is not equal to zero.

By assumption, $m\geq 2$. If $\gamma_0=\gamma_1=\gamma_2=0$, then $\beta_{1,m}=\beta_{2,m}=\beta_{3,m}=0$. By the definition of m at least one of the $\beta_{i,m}$ is not equal to zero. Hence the relation $(*)$ does not vanish identically, and the multiplicity of T at l is exactly m.

This proves Theorem 2.3.1 in Case 1.

REMARK 3.2.1. We also see that $\bar{b}^3$ is not a factor in $(*)$. This implies that a point $Q\in L$, where $Q\notin C$, does not correspond H-dually to a triple tangential direction (see §3.3).

3.3. In Cases 1, 2 and 3 we say that a j-fold tangential direction of T at l corresponds H-dually to the point $(0,0,z',w') \in L \subset P^3$ if $(w'\bar{a} + z'\bar{b})^j$ is a factor in the unique relation between $\bar{a}$ and $\bar{b}$ modulo $\mathcal{M}^{m+1}$. It is easy to see that for a point $Q \in L$ this property is independent of how we choose our coordinate system within the frame given in §3.1 (see §2.3).

PROOF OF THEOREM 2.3.2 IN CASE 1. Since $c \equiv d$ modulo $\mathcal{M}^2$, all the tangents to T at l are contained in $\check{H}$, the plane in G consisting of all lines in the special plane H. This proves (i).

For the proof of (ii) we may assume $Q = (0,0,0,1)$ without loss of generality. Then a (1-fold) tangential direction to T at l corresponds H-dually to Q iff the coefficient γ_m of $(*)$ is zero.

$$\gamma_m = \frac{\beta_{1,m}}{\alpha_{1,1}^m}(z_2 - z_3)z_1^m + \frac{\beta_{2,m}}{\alpha_{2,1}^m}(z_3 - z_1)z_2^m + \frac{\beta_{3,m}}{\alpha_{3,1}^m}(z_1 - z_2)z_3^m = 0$$

is equivalent to

$$\det \begin{bmatrix} \beta_{1,m}z_1^m & \alpha_{1,1}^m z_1 & \alpha_{1,1}^m \\ \beta_{2,m}z_2^m & \alpha_{2,1}^m z_2 & \alpha_{2,1}^m \\ \beta_{3,m}z_3^m & \alpha_{3,1}^m z_3 & \alpha_{3,1}^m \end{bmatrix} = 0.$$

Using the local parametrizations of C at the P_i in $L \cap C$ we see that this is equivalent to the existence of a triple (r_1, r_2, r_3) with $r_1 \neq 0$, such that the surface with affine equation

$$r_1 Y Z^m + r_2 X^m Z + r_3 X^m = 0$$

intersects C at least $m+1$ times at P_i for $i = 1,2,3$. On the other hand any surface satisfying property (c) of Theorem 2.3.2(ii) has affine equation

$$Y \cdot f(Z) + XY \cdot g(X,Y,Z) + X^m \cdot h(X,Z) + Y^2 i(Z) = 0$$

for polynomials f, g, h, i. By property (a) we can write $f(Z)$ as

$$r_0 Z^{m-1} + r_1 Z^m \quad \text{for scalars } r_0, r_1,$$

and we can write $h(X,Z)$ as

$$r_2 Z + r_3 + r_4 X \quad \text{for scalars } r_2, r_3, r_4.$$

By property (d) we have $r_0 = 0$, and then $r_1 \neq 0$ since $L \not\subset \mathrm{Sing}(M)$ by property (c).

Hence any surface M in question has affine equation

$$r_1 Y Z^m + r_2 X^m Z + r_3 X^m = 0 \quad \text{modulo}(XY, Y^2, X^{m+1}).$$

This completes the proof of Theorem 2.3.2.

3.4. PROOF OF THEOREM 2.3.3. Again we may assume $Q = (0,0,0,1)$. A 2-fold tangential direction of T at l corresponds H-dually to Q iff $\gamma_{m-1} = \gamma_m = 0$, i.e.

$$\frac{\beta_{1,m}}{\alpha_{1,1}^m}(z_2 - z_3)z_1^j + \frac{\beta_{2,m}}{\alpha_{2,1}^m}(z_3 - z_1)z_2^j + \frac{\beta_{3,m}}{\alpha_{3,1}^m}(z_1 - z_2)z_3^j = 0$$

for $j = m-1, m$. This is equivalent to

$$\frac{\beta_{1,m}}{\alpha_{1,1}^m} z_1^{m-1} = \frac{\beta_{2,m}}{\alpha_{2,1}^m} z_2^{m-1} = \frac{\beta_{3,m}}{\alpha_{3,1}^m} z_3^{m-1} \qquad (= r)$$

which is equivalent to the existence of a scalar r such that the surface with (affine) equation $YZ^{m-1} - rX^m = 0$ intersects C at least $m+1$ times at P_1, P_2, P_3. It is enough to show that any surface N satisfying (a), (b), (c) of Theorem 2.3.3 has equation $YZ^{m-1} - rX^m = 0$ modulo(XY, Y^2) for some $r \in K$. The equation of such a surface N is obviously homogeneous of degree m. The image of the function $X^j Z^{m-j}$ is of order j, for $j = 0, \ldots, m$, in $\mathcal{O}_{C,P_i}$, for each P_i different from Q. Hence no term of the form $X^j Z^{m-j}$ occurs in the equation of N for $j = 0, 1, \ldots, m-1$. Hence the equation of N is $r_1 YZ^{m-1} + r_2 X^m = 0 \bmod (XY, Y^2)$. Since $L \not\subset \operatorname{Sing}(N)$, $r_1 \neq 0$. Hence we have proved Theorem 2.3.3 in Case 1.

3.5. In Case 2 we use Weierstrass' Preparation Theorem to obtain the unique relation between $\bar{a}$ and $\bar{b}$ modulo $\mathcal{M}^{m+1}$:

$$(**) \qquad \det \begin{bmatrix} 1 & z_1 & f_1(\bar{a}, \bar{b}) \\ 0 & 1 & f_2(\bar{a}, \bar{b}) \\ 0 & z_2 & f_3(\bar{a}, \bar{b}) \end{bmatrix} = 0,$$

where

$$f_1(\bar{a}, \bar{b}) = \left(\frac{\bar{a} + z_1\bar{b}}{\alpha_{1,2}}\right)^m \cdot \beta_{1,2m},$$

$$f_2(\bar{a}, \bar{b}) = \left(\frac{\bar{a} + z_1\bar{b}}{\alpha_{1,2}}\right)^m \cdot \beta_{1,2m+1} + m \left(\frac{\bar{a} + z_1\bar{b}}{\alpha_{1,2}}\right)^{m-1} \cdot \left(\frac{\bar{b}}{\alpha_{1,2}} - \frac{\bar{a} + z_1\bar{b}}{\alpha_{1,2}^2} \cdot \alpha_{1,3}\right) \cdot \beta_{1,2m},$$

$$f_3(\bar{a}, \bar{b}) = \left(\frac{\bar{a} + z_2\bar{b}}{\alpha_{2,1}}\right)^m \cdot \beta_{2,m},$$

and where m is the number given in Theorem 2.3.1.

In Case 3 the corresponding relation is

$$(***) \qquad \begin{aligned} &\left(\frac{\bar{a} + z_1\bar{b}}{-\alpha_3}\right)^m \cdot \beta_{3m+2} + m \left(\frac{\bar{a} + z_1\bar{b}}{-\alpha_3}\right)^{m-1} \cdot \left(\frac{\alpha_4}{\alpha_3^2}(\bar{a} + z_1\bar{b}) - \frac{\bar{b}}{\alpha_3}\right) \cdot \beta_{3m+1} \\ &+ m \left(\frac{\bar{a} + z_1\bar{b}}{-\alpha_3}\right)^{m-1} \cdot \left(\frac{\alpha_5\alpha_3 - \alpha_4^2}{\alpha_3^2}(\bar{a} + z_1\bar{b}) + \frac{\alpha_4}{\alpha_3^2}\bar{b}\right) \cdot \beta_{3m} \\ &+ \binom{m}{2} \left(\frac{\bar{a} + z_1\bar{b}}{-\alpha_3}\right)^{m-2} \cdot \left(\frac{(\bar{a} + z_1\bar{b})}{\alpha_3^2} - \frac{\bar{b}}{\alpha_3}\right)^2 \cdot \beta_{3m} = 0. \end{aligned}$$

Treating $(**)$ and $(***)$ the same way as $(*)$ in Case 1, we obtain Theorems 2.3.1, 2.3.2 and 2.3.3 in Cases 2 and 3.

4. The last part of the proofs of the main results. We now consider an arbitrary point $l_x \in \tilde{T}$, reduced in its fibre over $l(L) \in T$. We keep the notation from §§2.1, 2.2 and 3.1, and assume that $\operatorname{rk}(\mathcal{O}_C \otimes_{\mathcal{O}_{P^3}} \mathcal{O}_L) = n \geq 3$. As before we identify points l_x in the fibre $\phi^{-1}(l(L))$ with k-tuples $(n_{1,x}, \ldots, n_{kx})$ such that $0 \leq n_{ix} \leq n_i$ for $i = 1, \ldots, k$, and $\sum_{i=1}^k n_{ix} = 3$, where $L \cap C = \{P_1, \ldots, P_k\}$ and $\operatorname{rk}(\mathcal{O}_C \otimes_{\mathcal{O}_{P^3}} \mathcal{O}_L)_{P_i} = n_i$.

To prove Theorems 2.3.1, 2.3.2 and 2.3.3 it is enough (see next Remark) to show

(a) The fibre $\tilde{T} \times_T \operatorname{Spec} K(l(L)) = \phi^{-1}(l(L))$ is reduced at l_x if and only if in the corresponding k-tuple $(n_{1x}, \dots, n_{kx})$ we have

$$n_{ix} \text{ is either } 0 \text{ or } n_i \quad \text{for } i = 1, \dots, k.$$

(b) If n_{ix} is either 0 or n_i for $i = 1, \dots, k$, then $\hat{\mathcal{O}}_{\tilde{T}, l_x} = R/\mathfrak{A}_x$, where $R = \hat{\mathcal{O}}_{G,l}$ and $\mathfrak{A}_x$ is the ideal in R generated by those T_i (see §3.1) such that $n_{ix} = n_i$.

REMARK. It is enough to show (a) and (b) because together these statements imply that the local geometry of $\tilde{T}$ at l_x is determined only by the local parametrizations of C at those P_i such that $n_{ix} = n_i \neq 0$, and it is determined in exactly the same way as "if L was a 3-secant only intersecting C in those points P_i where $n_{ix} = n_i \neq 0$".

The last is true because the coefficients of each separate T_i depend only on the local behaviour of C at P_i. Since $\sum_{i=1}^k n_{ix} = 3$, we are then in a situation already covered by the discussion in §3, i.e. in one of the Cases 1, 2, or 3.

PROOF OF (a) AND (b). We shall use the local description in [**GP**, pp. 19–23]. Consider the semilocal ring $\bigoplus_{i=1}^k A_i$, where $A_i = \hat{\mathcal{O}}_{C,(l,P_i)}$. We have $\bigoplus_{i=1}^k A_i = R[Z]/(S, D)$, where $S = \prod_{i=1}^k S_i$ and $D = \sum_{i=1}^k T_i \cdot \prod_{i \neq j} S_j$. In particular D is a polynomial of degree $n-1$ in Z, with coefficients in $(a, b, c, d) \cdot K[[a, b, c, d]]$. (In [**GP**] one denotes this polynomial by T.)

Let B be the semilocal ring obtained by blowing up $F^3(\bigoplus_{i=1}^k A_i)\hat{\mathcal{O}}_{T,l}$ in $\hat{\mathcal{O}}_{T,l}$. It is shown in [**GP**, Proposition 3.3], that

$$B = R[X_0, \dots, X_{n-4}]/(V_0, V_1, V_2, r_0, \dots, r_{n-4}),$$

where $r_0 + r_1 Z + \cdots + r_{n-4} Z^{n-4}$ is the rest of the Euclidean division of S by $\Omega = Z^{n-3} + X_{n-4} Z^{n-4} + \cdots + X_1 Z + X_0$.

We now describe the V_i: For an element P of $R[X_0, \dots, X_{n-4}][Z]$ denote by $\overline{P}^S$ its image in

$$R[X_0, \dots, X_{n-4}][Z]/(S).$$

The $R[X_0, \dots, X_{n-4}]$-module $R[X_0, \dots, X_{n-4}][Z]/(S)$ is free of rank n, and it is generated by $\overline{Z^j}^S$, $j = 0, \dots, n-1$. Let

$$\tau \in \operatorname{Hom}_{R[X_0, \dots, X_{n-4}]}(R[X_0, \dots, X_{n-4}][Z]/(S), R[X_0, \dots, X_{n-4}])$$

be defined by

$$\tau(\overline{Z^j}^S) = \delta_{j,n-1} \quad \text{(Kronecker delta function)}$$

for $j = 0, \dots, n-1$. Then we have $V_i = \tau(\overline{D\Omega Z^i}^S)$ for $i = 0, 1, 2$. The maximal ideals of B correspond to those $(x_0, \dots, x_{n-4}) \in K^{n-3}$ such that $\Omega_x = x_0 + x_1 Z + \cdots + x_{n-4} Z^{n-4} + Z^{n-3}$ divides $\prod_{i=1}^n (Z - z_i)^{n_i}$. For such an $(n-3)$-tuple $(x_0, \dots, x_{n-4})$ the maximal ideal $\mathfrak{n}_x$ is the image of $(a, b, c, d, X_0 - x_0, \dots, X_{n-4} - x_{n-4})$ in B. When $\Omega_x = \prod_{i=1}^k (Z - z_i)^{n_i - n_{ix}}$, the k-tuple corresponding to l_x is $(n_{1x}, \dots, n_{kx})$.

Denote by $\bar{r}_j$ the image of r_j in $R[X_0, \dots, X_{n-4}]/(a, b, c, d)$, $j = 0, \dots, n-4$. By construction each r_j is contained in $(X_0 - x_0, \dots, X_{n-4} - x_{n-4})$. Since the coefficients of D are nonunits in R, each V_i is a nonunit in R, $i = 0, 1, 2$. This implies that the fibre $\phi^{-1}(l(L))$ is reduced at l_x if and only if the images of $\bar{r}_0, \ldots, \bar{r}_{n-4}$ span the vector space

$$(X_0 - x_0, \ldots, X_{n-4} - x_{n-4})/(X_0 - x_0, \ldots, X_{n-4} - x_{n-4})^2.$$

By Lemma 3.5 of [**GP**, p. 22], this happens if and only if the polynomials

$$\prod_{i=1}^{k}(Z - z_i)^{n_i} \quad \text{and} \quad \prod_{i=1}^{k}(Z - z_i)^{n_i - n_{ix}}$$

have no common factor, i.e. n_{ix} is either 0 or n_i, $i = 1, \ldots, k$. This proves (a).

Under the assumption of (b) this lemma also gives

$R \cong R_x$, where R_x is the ring
$[R[X_0, \ldots, X_{n-4}]/(r_0, r_1, \ldots, r_{n-4})]$ localized in the ideal
$(a, b, c, d, X_0 - x_0, \ldots, X_{n-4} - x_{n-4})$.

Hence $B_{\mathfrak{n}_x} \simeq R_x/(V_0, V_1, V_2)$. Since R_x is a regular local ring, $R_x[Z]$ is a UFD. Identifying all polynomials with their images in $R_x[Z]$, we have in $R_x[Z]$

$$S = S'S'' = \Omega\Delta,$$

where $S' = \prod_{i, n_{ix}=0} S_i$, $S'' = \prod_{i, n_{ix} \neq 0} S_i$, and Δ is some power series. Since Ω_x and $\Delta_x = \prod_i (Z - z_i)^{n_{ix}}$ have no common factors, we obtain $S' = \Omega$ in $R_x[Z]$. Hence $V_i = \tau(\overline{DS'Z^i})$, $i = 0, 1, 2$, and one easily obtains that (V_0, V_1, V_2) generates the same ideal as the coefficients of those T_i where $n_{ix} \neq 0$. Since $\sum_{i=1}^{k} n_{ix}$, we are in a situation already covered by the discussion in §3.

A more detailed proof of (a) and (b) is contained in [**J**].

5. Examples and remarks. Let C be a smooth complete intersection of two cubic surfaces. Then C contains no line since a complete intersection is connected, and since a reducible connected curve is singular.

Furthermore C possesses no 4-secants since a 4-secant would have been contained in both cubic surfaces by Bezout's theorem, and then C would have contained a line. Assume L is a 3-secant corresponding to a point $l \in T$.

As a consequence of Proposition 2.2.1, the following three statements are equivalent:

(a) T is singular at l.

(b) There exists a plane H and a cubic surface F_3 containing C such that $H \cap F \supseteq L_2$, where L_2 is the double line in H with support L.

(c) There exists a cubic surface F_3 containing C and having two (possibly coinciding) singularities on L.

If T is singular at l, the surface F_3 of (b) is the same as the surface F_3 of (c).

Furthermore, as a consequence of the results in §2, we have: The multiplicity of T at the point l is 3 iff there exists a plane H and a cubic surface F_3 containing C such that $H \cap F_3 = L_3$, where L_3 is the triple line in H with support L. The multiplicity of T at any point l is at most 3.

Assume that the multiplicity of T at l is 2. Then l is a nonordinary singularity on T iff $H \cap F = L_2 \cup L'$, where L' intersects L in one of the singular points of F_3 on L. Then the unique double tangential direction of T at l corresponds H-dually to the other singular point of F_3 on L.

If the two singularities of F_3 on L coincide at a point Q, but $L' \cap L \neq \{Q\}$, then a single tangential direction corresponds H-dually to Q.

Assume the multiplicity of T at l is 3. Then l is a nonordinary singularity on T iff the two singular points of F_3 on L coincide. In this case a double, but not a triple tangential direction of T at l corresponds H-dually to the unique singular point of F_3 on L.

REMARKS. (1) If we study "the initial forms" $(*)$, $(**)$ and $(***)$ of §3 more closely, we can say more about the local geometry of $\tilde{T}$ at a point l_x whose fibre $\phi^{-1}(l(L))$ is reduced at l_x. When $m \geq 3$, we have:

The singularity is totally nonordinary, i.e. all tangential directions coincide iff

$$\sum_{i, n_{ix} = n_i} [(m+1)n_{ix} - \min(m_i, (m+1)n_{ix})] = 1,$$

where $m_i = I(P_i, C \cap H)$ for the special plane H. Then the point corresponding H-dually to the m-fold tangential direction is the unique $P_i \in L \cap C$ such that $m_i = (m+1)n_{ix} - 1$.

(2) Using essentially the same methods as in the proofs of the results in §2.3 we may examine the local geometry of T at nonsingular points of T. Let $\mathcal{L}$ be the tangent line to T at a nonsingular point $l(L)$. Then we have, for example, l is a flex on T, i.e. $\operatorname{rk}(\mathcal{O}_T \otimes_{\mathcal{O}_{P^5}} \mathcal{O}_{\mathcal{L}})_l \geq 3$, iff $L \cap C = \{P_1, P_2, P_3\}$, $n_1 = n_2 = n_3 = 1$, and $I(P_1, C \cap H) = 1$, $I(P_j, C \cap H) \geq 3$, $j = 2, 3$, for some plane H, or: $L \cap C = \{P_1, P_2\}$, $n_1 = 1$, $n_2 = 2$, and $I(P_1, C \cap H) = 1$, $I(P_2, C \cap H) \geq 6$, for some plane H.

(3) When the fibre $\phi^{-1}(l)$ is not reduced at a point $l_x \in \tilde{T}$, the multiplicity formula given in Theorem 2.3.1 is no longer valid, although Proposition 2.3.1 holds. In addition the dimension of the tangent space of $\tilde{T}$ at l_x may be 3.

As mentioned in §1 there is a (set-theoretical) bijection between $\tilde{T}$ and the curve $\mathrm{Sec}_3(C)$ defined in [**La1**]. In [**WL**] one shows that the same multiplicity formula as given in our Theorem 2.3.1 holds for points of $\mathrm{Sec}_3(C)$ corresponding to the points of $\tilde{T}$, at which their respective fibres of the blowing-up map $\phi\colon \tilde{T} \to T$ are reduced. In fact the initial forms are essentially the same.

It is an open question whether the same is true for the remaining points (if any) of $\mathrm{Sec}_3(C)$ and $\tilde{T}$.

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